# Some remarks on grand Furuta inequality (Inequalities on Linear Operators and its Applications)



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# Some remarks on grand Furuta inequality

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**1. Introduction.** Throughout this note, $A$ and $B$ are positive operators on a Hilbert space. For convenience, we denote $A \geq 0$ (resp. $A > 0$) if $A$ is a positive (resp. invertible) operator. We begin from Furuta inequality ([6],[7],[9]).

**Furuta inequality:** *If $A \geq B \geq 0$, then for each $r \geq 0$,*

(F) $$A^{\frac{p+r}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}} \quad \text{and} \quad (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq B^{\frac{p+r}{q}}$$

*holds for $p$ and $q$ such that $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.*

This yields the Löwner-Heinz inequality;

(LH) $$A \geq B \geq 0 \ \text{ implies } \ A^{\alpha} \geq B^{\alpha} \ \text{ for any } \ \alpha \in [0,\ 1].$$

We had reformed (F) in terms of the $\alpha$-power mean (or generalized geometric operator mean) of $A$ and $B$ which is introduced by Kubo-Ando as follows [16]:

$$A \,\sharp_{\alpha}\, B = A^{\frac{1}{2}}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\alpha} A^{\frac{1}{2}} \quad \text{for } \ \alpha \in [0,\ 1],$$

the case $\alpha \notin [0,\ 1]$, we use the notation $\natural$ to distinguish the operator mean.

By using the $\alpha$-power mean, Furuta inequality is given as follows:

(F) $$A \geq B \geq 0 \ \text{ implies } \ A^{-r} \,\sharp_{\frac{1+r}{p+r}}\, B^p \leq A \ \text{ for } \ p \geq 1 \ \text{ and } \ r \geq 0.$$

Based on this reformulation, we had proposed a satellite form of (F) [12],[13];

(SF) $$A \geq B \geq 0 \ \text{ implies } \ A^{-r} \,\sharp_{\frac{1+r}{p+r}}\, B^p \leq B \leq A \ \text{ for } \ p \geq 1 \ \text{ and } \ r \geq 0.$$

On the other hand, Ando and Hiai showed the next inequality [1],[11].

**Ando-Hiai inequality:** Ando-Hiai had shown the following inequality:

(AH) *If* $A \,\sharp_{\alpha}\, B \leq I$ *for* $A,\ B > 0$, *then* $A^r \,\sharp_{\alpha}\, B^r \leq 1$ *holds for* $r \geq 1$.

From this relation, they had shown the following inequality ($\mathrm{AH}_0$). It is equivalent to the main result of log majorization and can be given as the following form:

($\mathrm{AH}_0$) $$A^{-1} \,\sharp_{\frac{1}{p}}\, A^{-\frac{1}{2}} B^p A^{-\frac{1}{2}} \leq I \ \Rightarrow \ A^{-r} \,\sharp_{\frac{1}{p}}\, (A^{-\frac{1}{2}} B^p A^{-\frac{1}{2}})^r \leq I, \ \ p \geq 1,\ r \geq 1.$$

Furuta had constructed the following inequality which interpolats ($AH_0$) and (F), we call this grand Furuta inequality ([2],[4],[8],[9]).

**Grand Furuta inequality:** *If $A \geq B \geq 0$ and $A > 0$, then for each $1 \leq p$ and $t \in [0,\ 1]$,*

$$\text{(GF)} \qquad A^{-r} \natural_{\frac{1-t+r}{(p-t)s+r}} (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^s \leq A^{1-t}$$

*holds for $t \leq r$ and $1 \leq s$.*

The satellite form of (GF) is given also as follows ([2],[14]):

$$\text{(SGF)} \qquad A^{-r+t} \natural_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \leq B \ (\leq A).$$

We pointed out that (F) and (AH) are obtained from each other and gave a generarized form of (AH) ([3],[5]).

*For $\alpha \in (0,\ 1)$ fixed,*

$$\text{(GAH)} \qquad A \natural_\alpha B \leq I \implies A^r \natural_{\frac{\alpha r}{(1-\alpha)s+\alpha r}} B^s \leq I \ \text{ for } r,\ s \geq 1.$$

Using (GAF), we modified (GF) as follows [15]:

**Theorem A.** *If $A \geq B \geq 0$ and $A > 0$, then for each $1 \leq p$ and $t \in [0,\ 1]$,*

$$A^{-r+t} \natural_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \leq A^t \natural_{\frac{1-t}{p-t}} B^p$$

*holds for $t \leq r$ and $1 \leq s$.*

Recently, Furuta has shown the following theorem concerning to the above theorem [10].

**Theorem F.** *Let $A \geq B \geq 0$ with $A > 0$, $t \in [0,\ 1]$ and $p \geq 1$. Then*

$$F(\lambda,\ \mu) = A^{-\frac{\lambda}{2}} \{A^{\frac{\lambda}{2}} (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^\mu A^{\frac{\lambda}{2}}\}^{\frac{1-t+\lambda}{(p-t)\mu+\lambda}} A^{-\frac{\lambda}{2}}$$

*satisfies the following properties:*

$$\text{(i)} \qquad F(r,\ w) \geq F(r,\ 1) \geq F(r,\ s) \geq F(r,\ s')$$

*holds for any $s' \geq s \geq 1,\ r \geq t$ and $1 - t \leq (p-t)w \leq p - t$.*

$$\text{(ii)} \qquad F(q,\ s) \geq F(t,\ s) \geq F(r,\ s) \geq F(r',\ s)$$

*holds for any* $r' \geq r \geq t,\ s \geq 1$ *and* $t-1 \leq q \leq t$.

In this note, we observe this theorem from the $\alpha$−power mean.

**2. Review of Theorem F.** We rewrite Theorem F by the form of $\alpha$-power mean. Then

$$F(\lambda,\ \mu) = A^{-\lambda} \ \sharp_{\frac{1-t+\lambda}{(p-t)\mu+\lambda}} \ (A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^{\mu}$$

and by putting $B_1 = (A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^{\frac{1}{p-t}}$, (i) and (ii) of Theorem F are written as follows:

(i)
$$A^{-r} \ \sharp_{\frac{1-t+r}{(p-t)w+r}} \ B_1^{(p-t)w} \geq A^{-r} \ \sharp_{\frac{1-t+r}{p-t+r}} \ B_1^{p-t}$$
$$\geq A^{-r} \ \sharp_{\frac{1-t+r}{(p-t)s+r}} \ B_1^{(p-t)s} \geq A^{-r} \ \sharp_{\frac{1-t+r}{(p-t)s'+r}} \ B_1^{(p-t)s'}$$

and

(ii)
$$A^{-q} \ \sharp_{\frac{1-t+q}{(p-t)s+q}} \ B_1^{(p-t)s} \geq A^{-t} \ \sharp_{\frac{1}{(p-t)s+t}} \ B_1^{(p-t)s}$$
$$\geq A^{-r} \ \sharp_{\frac{1-t+r}{(p-t)s+r}} \ B_1^{(p-t)s} \geq A^{-r'} \ \sharp_{\frac{1-t+r'}{(p-t)s+r'}} \ B_1^{(p-t)s}.$$

We point out that Theorem A can be written more precisely,

$$A^{-r+t} \ \sharp_{\frac{1-t+r}{(p-t)s+r}} \ (A^t \ \natural_s \ B^p) \leq (A^t \ \natural_s \ B^p)^{\frac{1}{(p-t)s+t}} \leq B \leq A^t \ \sharp_{\frac{1-t}{p-t}} \ B^p.$$

Because $A^{-r+t} \ \sharp_{\frac{1-t+r}{(p-t)s+r}} \ (A^t \ \natural_s \ B^p) \leq (A^t \ \natural_s \ B^p)^{\frac{1}{(p-t)s+t}} \leq B$ is already shown in our proof of (SGF). So the result of Theorem A has shown the folloing inequality.

$$A^{-r} \ \sharp_{\frac{1-t+r}{(p-t)s+r}} \ B_1^{(p-t)s} \leq A^{-t} \ \sharp_{\frac{1}{(p-t)s+t}} \ B_1^{(p-t)s} \leq B_1^{1-t},$$

and Furuta improved on the second inequality of this form to

$$A^{-t} \ \sharp_{\frac{1}{(p-t)s+t}} \ B_1^{(p-t)s} \leq A^{-q} \ \sharp_{\frac{1-t+q}{(p-t)s+q}} \ B_1^{(p-t)s}, \quad t-1 \leq q \leq t.$$

Furuta's process is the following:
Since $0 \leq t-q \leq 1$, $(A^{\frac{t}{2}}B_1^{(p-t)s}A^{\frac{t}{2}})^{\frac{t-q}{(p-t)s+t}} \leq A^{t-q}$ holds by (LH), and we can obtain the result as follows:

$$\begin{aligned} & A^{-t} \ \sharp_{\frac{1}{(p-t)s+t}} \ B_1^{(p-t)s} \\ = \ & B_1^{(p-t)s} \ \sharp_{\frac{(p-t)s-1+t}{(p-t)s+t}} \ A^{-t} \\ = \ & B_1^{(p-t)s} \ \sharp_{\frac{(p-t)s-1+t}{(p-t)s+q}} \ (B_1^{(p-t)s} \ \sharp_{\frac{(p-t)s+q}{(p-t)s+t}} \ A^{-t}) \end{aligned}$$

$$
\begin{aligned}
&= B_1^{(p-t)s} \natural_{\frac{(p-t)s-1+t}{(p-t)s+q}} (A^{-t} \natural_{\frac{t-q}{(p-t)s+t}} B_1^{(p-t)s}) \\
&= B_1^{(p-t)s} \natural_{\frac{(p-t)s-1+t}{(p-t)s+q}} A^{-\frac{t}{2}}(A^{\frac{t}{2}} B_1^{(p-t)s} A^{\frac{t}{2}})^{\frac{t-q}{(p-t)s+t}} A^{-\frac{t}{2}} \\
&\le B_1^{(p-t)s} \natural_{\frac{(p-t)s-1+t}{(p-t)s+q}} A^{-\frac{t}{2}} A^{t-q} A^{-\frac{t}{2}} \\
&= A^{-q} \natural_{\frac{1-t+q}{(p-t)s+q}} B_1^{(p-t)s}.
\end{aligned}
$$

**3. Modification of Theorem F.** Furuta's results (i) and (ii) are holds suppose $A \ge B_1$, but in Theorem F this order does not hold. We search a suitable relation between $A$ and $B_1$ by the help of (GAH).

$A \ge B \ge 0$ implies $A^t \ge B^t \ge 0$ for $t \in [0,\ 1]$ by (LH). This is equivalent to $A^{-t} \natural_{\frac{t}{p}} B_1^{p-t} \le I$. By (GAH), we have

$$A^{-r} \natural_{\frac{r}{p-t+r}} B_1^{(p-t)} = B_1^{(p-t)} \natural_{\frac{p-t}{p-t+r}} A^{-r} \le I.$$

That is,

$$A \ge B \ge 0 \ \Rightarrow\ A^{-r} \natural_{\frac{r}{p-t+r}} B_1^{(p-t)} \le I \ \Rightarrow\ A^{-r'} \natural_{\frac{r'}{(p-t)s+r'}} B_1^{(p-t)s} \le I \text{ for } r' \ge r,\ s \ge 1.$$

So we begin from the assumption $A^{-r} \natural_{\frac{r}{p-t+r}} B_1^{(p-t)} \le I$.

**Lemma 1.** *Let $A,\ B \ge 0$ and $A^{-r} \natural_{\frac{r}{p+r}} B^p \le I$ for $p,\ r \ge 0$. Then the following hold:*

(i) $$A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p \le B^\delta \qquad 0 \le \delta \le p$$

*and*

(ii) $$A^{-r} \natural_{\frac{\lambda+r}{p+r}} B^p \le A^\lambda \qquad -r \le \lambda \le 0.$$

These results are already known, but these play essential roles in our following discussions. We can arrange Theorem F as follows except $F(q,\ s) \ge F(t,\ s)$ for $t-1 \le q \le 0$.

**Theorem 1.** *Let $A,\ B \ge$ and $A^{-r} \natural_{\frac{r}{p+r}} B^p \le I$ for $p,\ r \ge 0$. Then*

(1) $$A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p \le A^{-r} \natural_{\frac{\delta+r}{\mu+r}} B^\mu$$

*holds for $p \ge \mu \ge \delta \ge 0$ and*

(ii) $$A^{-r} \natural_{\frac{\lambda+r}{p+r}} B^p \le A^{-t} \natural_{\frac{\lambda+t}{p+t}} B^p$$

*holds for* $r \geq t \geq 0, \quad -t \leq \lambda \leq p.$

**Proof.** (i) is obtained by the followin calculation:

$$A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p = A^{-r} \natural_{\frac{\delta+r}{\mu+r}} (A^{-r} \natural_{\frac{\mu+r}{p+r}} B^p) \leq A^{-r} \natural_{\frac{\delta+r}{\mu+r}} B^{\mu}.$$

(ii) can be shown as follows:

$$\begin{aligned} A^{-r} \natural_{\frac{\lambda+r}{p+r}} B^p &= B^p \natural_{\frac{p-\lambda}{p+r}} A^{-r} = B^p \natural_{\frac{p-\lambda}{p+t}} (B^p \natural_{\frac{p+t}{p+r}} A^{-r}) \\ &= B^p \natural_{\frac{p-\lambda}{p+t}} (A^{-r} \natural_{\frac{-t+r}{p+r}} B^p) \leq B^p \natural_{\frac{p-\lambda}{p+t}} A^{-t} = A^{-t} \natural_{\frac{\lambda+t}{p+t}} B^p. \end{aligned}$$

**4. Applications.** Return to Theorem A, we summarize the above discussions.

**Theorem A(1).** *If* $A \geq B \geq 0$ *and* $t \in [0,\ 1]$, $p \geq t,\ r \geq t,\ 0 \leq \delta \leq (p-t)s,$ *then*

$$A^{-r+t} \natural_{\frac{\delta+r}{(p-t)s+r}} (A^t \natural_s B^p) \leq (A^t \natural_s B^p)^{\frac{\delta+t}{(p-t)s+t}} \leq A^{\alpha} \natural_{\frac{\delta+t-\alpha}{(p-t)s+t-\alpha}} (A^t \natural_s B^p)$$

*holds for* $\min\{\delta+t,\ 1\} \geq \alpha \geq 0.$

This is equivalent to

$$A^{-r} \natural_{\frac{\delta+r}{(p-t)s+r}} B_1^{(p-t)s} \leq A^{-t} \natural_{\frac{\delta+t}{(p-t)s+t}} B_1^{(p-t)s} \leq A^{\alpha-t} \natural_{\frac{\delta+t-\alpha}{(p-t)s+t-\alpha}} B_1^{(p-t)s}.$$

If $p \geq 1$ and $\delta = 1-t, \quad \alpha = t-q$, we have Furta's result (ii) containing the case $t-1 \leq q \leq 0.$

Under the assumption $A^{-r} \natural_{\frac{r}{p-t+r}} B_1^{(p-t)} \leq I$, our Theorem A can be written as follows:

**Theorem A(2).** *Let* $A,\ B \geq 0$ *and put* $B_1 = (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^{\frac{1}{p-t}}$ *for* $p \geq t \geq 0$. *If* $A^{-r} \natural_{\frac{r}{p-t+r}} B_1^{(p-t)} \leq I$ *for* $r \geq t \geq 0$, *then for* $s \geq 1$

(i) $$A^{-r+t} \natural_{\frac{\delta+r}{(p-t)s+r}} (A^t \natural_s B^p) \leq A^{-r+t} \natural_{\frac{\delta+r}{\mu+r}} (A^t \natural_{\frac{\mu}{p-t}} B^p)$$

*holds for* $0 \leq \delta \leq \mu \leq (p-t)s$ *and*

(ii) $$A^{-r+t} \natural_{\frac{\lambda+r}{(p-t)s+r}} (A^t \natural_s B^p) \leq (A^t \natural_s B^p)^{\frac{\lambda+t}{(p-t)s+t}}$$

*holds for* $-t \leq \lambda \leq (p-t)s.$

But this case reduces to Theorem 1 because

$$\text{(i)} \iff A^{-r} \natural_{\frac{\delta+r}{(p-t)s+r}} B_1^{(p-t)s} \leq A^{-r} \natural_{\frac{\delta+r}{\mu+r}} B_1^{\mu}$$

and

$$\text{(ii)} \iff A^{-r} \natural_{\frac{\lambda+r}{(p-t)s+r}} B_1^{(p-t)s} \leq A^{-t} \natural_{\frac{\lambda+t}{(p-t)s+t}} B_1^{(p-t)s}.$$

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